Will we ever find a circle?

Do mathematical objects exist in the same way as physical objects?

By Jamie Freestone

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[1]The philosophy of mathematics may seem a very dry subject or, at best, a highly abstract and esoteric one. But investigations into the basic philosophical questions concerning mathematics actually intertwine with the last century's most important questions in philosophy more broadly and go to the heart of how we perceive the world in a scientifically advanced era. Additionally, any good survey of the developments in this field[2] will expose an incredibly rich panoply of ideas, speculations, mystic visions, theorems of gargantuan consequence, romantic flashes of genius and the presence in the world of real magic. We will attempt here merely to sketch an outline of the debate which concerns the central questions in the philosophy of mathematics, namely: do mathematical objects exist? and if so do they exist in the same way as do physical objects? Hopefully such an outline will provide a summary of the main positions in the contemporary debate and an indicative glimpse of the astonishing range of ideas and stretches of the imagination characteristic of modern mathematics itself.

The origins of this debate are as old as any in philosophy. Pythagoras, the pre-socratic philosopher, mathematician and cult-leader, believed that numbers were divine, that reality was ultimately comprised of numbers and that this numerical ontology [3] was primary over the comparatively maculate world of appearance and materiality. Plato considered mathematics (especially geometry) to be the highest calling, debarring entry into his philosophical academy to anyone not versed in mathematics. Indeed Plato's theory of Forms held that there was an ideal, timeless world of perfect objects compared to which the objects that we see are imperfect copies — an idea that developed directly out of the Pythagorean stance that numbers were divine.

In Europe[4] during the Renaissance, Galileo said that the book of nature is written in the language of mathematics and set in motion an attitude towards numbers shared by Newton and subsequent practitioners, which held that numbers are perfect, irreducible, immutable and eternal, but that the primary role of mathematics was to accurately

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[1] Footnotes will contain references but also personal comments, and they (these) will not be (aren't) written in academic prose. The body text will be accessible and hopefully clear, so feel free to just read the body; footnotes are for those who want further reading, tidbits and sundry mathematical arcana.

[2] For the history of Western mathematics, Carl Boyer and Morris Kline can hook you up. They're still the best historical works. Both are beautifully written and accessible to non-mathematicians like me. For the philosophy of maths, it gets trickier, but Stewart Shapiro is pretty good and certainly up to date. For a primer (a level up from this) see the relevant *Stanford Encyclopedia of Philosophy* entry by Horsten.

[3] A philosopher's word for the study of the basics of being, existence and reality.

[4] It is annoyingly difficult to get information (in English) on the development of these ideas in Arabic, Persian, Vedic or Chinese mathematics.

describe the physical world; this limited view no doubt emanated partly from the the ignorance of higher mathematics, particularly the mathematics of infinity. By the eighteenth century mathematics had become an extremely powerful set of practical applications, especially in utilising calculus as developed by Leibniz and Newton. The French mathematicians of the revolutionary and Napoleonic era were immensely confident in the ability of mathematics (mingled with mechanics, astronomy and the nascent discipline of engineering) to accurately describe the processes of the physical world. The apotheosis of this conviction was certainly Pierre Simon de Laplace who remarked to Napoleon that his new model of the solar system was perfectly described by the laws of motion and the mathematics of calculus, without any need for divine intervention; he further claimed that if a mathematician was supplied with all the relevant inputs of a physical system, then they could predict the outputs and final state and that in principle this should apply to the universe as a whole, with the future yielding new sciences that would predict all aspects of nature and human behaviour with precision.

This confidence was shattered by several developments in the nineteenth century. The discovery of non-Euclidean geometry showed that entirely new geometries could be devised that seemed to describe space in entirely unintuitive ways, but which were more rigorously defined and mathematically consistent than standard Euclidean geometry[5]. Work on infinity led to the definition of *transfinite* numbers, which established degrees of infinitude with some infinities apparently being larger than others[6]. This was another

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[5]That's a whole other story. In brief, Hungarian maths guy Farkas Bolyai worked for a long time on trying to sure-up Euclid's wayward parallel line postulate (a popular but fruitless mathematical pursuit of the time), warning his geometer son not to waste his time on the same: "You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy from my life. I entreat you, leave the science of parallels alone…I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours; my creations are far better than those of others and yet I have not achieved complete satisfaction. For here it is true that *si paullum a summo discessit, vergit ad imum*. I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind… I have travelled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness — *aut Caesar aut nihil*."

Undeterred, the firebrand Janos wrote back to his father to announce that he had circumvented Euclid's parallel line postulate by ignoring it all together and instead developing a whole new geometry, a whole new approach to space that was mathematically elegant: "Out of nothing I have created a strange new world!" For a great tale of this and other explosions of 19th C. maths, see *Duel at Dawn* by Amir Alexander; it's badly written so not in the Bibliography. Kline & Morris have relevant chapters.

[6] This is an extraordinary field. Most of the pioneering work was done by the German Georg Cantor (1845–1918), a poor man who was mocked ceaselessly for his ideas, who had to hide his Jewish ancestry and suffered from severe depression throughout his life. David Foster Wallace wrote an almost forgotten book about the bracing intellectual battle over the concept of the infinite (*q.v.* Bibliography). It's the only book I've ever found that is written in completely engaging language but which doesn't compromise any technical details (unlike this essay); it's also the only book on any serious topic written in exactly the voice of the author, David Foster Wallace, complete with elaborate footnotes (like this

difficult concept that nonetheless made perfect mathematical sense. Work in abstract algebra then established that solutions to certain equations of higher degree[7] can be solved in principle, without needing to be solved individually; in other words one can determine whether an entire class of equations is solvable by conducting operations on the algebraic structure of the equation, rather than working deductively to find a solution for a particular equation. In so doing, various abstract structures were discovered, such as groups and rings[8], which operate on a higher level than equations themselves, but which obey mathematical rules[9]. Together, these discoveries suggested that mathematics was not simply a language for describing physical processes like the cycles of planets, distances in navigation, or tensile strength in bridge building. Rather it had become obvious that mathematics was far more abstract, far more idealised and perhaps far more vast than the physical world. Transfinite numbers, non-Euclidean objects, algebraic rings: to the mathematicians and philosophers at the start of the twentieth century, these strange entities seemed entirely otherworldly, perhaps even platonic and yet the intellectual mood at the time was one of steadfast scientific empiricism, intolerant to anything hinting at mysticism or transcendence.

This is where the great crisis in the philosophy of mathematics originates and we find the immediate splintering into three schools of thought that we will deal with in turn [10]. Note also that because the crisis was produced directly from the vanguard of professional mathematics, the key figures in the three schools are generally mathematicians rather than philosophers.

essay), interesting asides, admissions of ignorance and a pleasing blend of precise technical language and warm informalities.

[7] Any equation involving terms raised to higher powers, *e.g.* squared, cubed or higher powers.

[8] These sound simple — they are not.

[9] Evariste Galois (1811–32) was the exemplar of the heroic mathematicians of the Romantic epoch. He died in a duel over the woman he loved, penning his final paper on group theory the night before he died —alas this story is at least somewhat apocryphal.

[10] Please don't be deterred by all the isms that are coming up. I'll try and keep it to a minimum but that's what you get when philosophers name things. *Duel at Dawn* also covers this but, again, it's poorly written: Kline and Boyer do it better.

Logicism is mainly associated with Bertrand Russell[11], who was a philosopher trained in mathematics who specialised in the intersection of the two disciplines. He was convinced of two things: that there needed to be a consolidation of the foundations of mathematics and that the foundation would be logic. He began working[12] on what was designed to be a rigorously argued, comprehensive, theoretical basis for all future mathematical endeavours, with the appropriately lofty title, *Principia Mathematica*[13] (hereafter PM). The task was enormous and successive revisions and expansions resulted in Russell dedicating almost ten years, eight hours a day, of constant study[14] to the task between 1903 and 1913. The result was a work that took over 300 pages to rigorously prove, with recourse to a set of basic axioms and rules for logical deduction, that 1+1=2. Unfortunately, it ultimately failed to do so (see below).

Russell's lasting legacy was a way to define the natural numbers (positive integers, or "counting" numbers) in terms of sets; a set is simply a collection of elements, such as the set of all even numbers, the set of numbers greater than one billion, the set of all people named Condoleezza, *etc*. The number two can be defined as a commonality between all sets which have one more than one element in them. From there, one can define all the natural numbers and through some more complicated steps define the real numbers[15] and then perform arithmetic. The third volume of PM stopped here, but it was implied by the authors — and accepted by mathematicians — that, in principle, if one could form a logical, axiomatic basis for arithmetic and numbers then all higher mathematics could be built up from that, albeit laboriously.

Russell also invented an eponymous paradox, which he thought he had solved, but which would return later as one of the bizarre flaws present in all logical systems. His paradox is best relayed in parable form: in a remote rural village in Italy all men are clean

[11] Who is an absolute dude. Not only was he a world famous mathematician and philosopher by the time he was 30, he was also the leading atheist in Europe at a time when it was OK for intellectuals to abstain from religious belief but not to openly criticise Christianity. He went on to be the only notable public figure to oppose WWI from the start, almost singlehandedly founding the modern pacifist movement; hitherto, pacifism was the preserve of Quakers and communists only. He also championed sexual liberation well before it was cool and then following WWII he was instrumental in the anti-nuclear movement. He was also very intellectually honest and whenever his work was superseded he admitted as much. Largely indestructible, he lived to age 98 (1872–1970) including surviving a plane crash off the coast of Denmark in his 70s, when he simply swam to shore, when many others didn't make it. There's a great clip online from an interview for early 60s TV. The 80-something Russell is being asked by an incredulous American interviewer why he doesn't believe in God. Russell (who sounds like — and at this stage of life, looks like — a wizard) doesn't miss a beat: "Because I've examined the stock arguments in favour of the existence of God and found them all to be logically invalid." Amen.

[12] Along with his mentor at Cambridge, Alfred North Whitehead (1861–1947).

[13] The title alludes to Newton's *Philosophiae Naturalis Principia Mathematica*, the seminal work of modern physics and was intended to have an equivalent impact in mathematics and logic.

[14] Russell later commented that his capacity for abstract thought was ruined: thereafter he focused on education, history, politics, activism and never returned to mathematics or logic, despite living another six decades.

[15] The set of real numbers includes all the natural numbers plus all fractions, negative numbers and irrational numbers like π or the square root of two.

shaven and there is only one barber, who will only shave the beards of men who don't shave themselves; does the barber shave himself? Whether he shaves his beard or not he is in violation of his own rule. When we move this across to set theory, it is highly problematic because sets can contain other sets: the set of all things beginning with S will contain various numbers, objects and some sets; sets can even contain themselves: the set of all things that aren't the letter S will contain this selfsame set. So what about the set of all sets that *don't* contain themselves? Would this set fit into itself? If it does, it violates its own rule and so we can see that the paradox is of the same logical form as the barber example, though much sillier. Nevertheless it terrified the mathematicians and logicians of pre-WWI Europe like nothing else could. Russell circumvented it by developing the theory of types, which said that a set was a higher level than an element and therefore a different type and that types should not mix, meaning that sets could only contain elements, not other sets. Thus, with a hierarchy of types in place, he proceeded with the extremely arduous task of showing why 1+1=2. [16]

Meanwhile *intuitionists* offered a somewhat more daring explanation for what mathematics is by declaring that we have an evolved, intuitive understanding of the natural numbers and from these we are able to build more elaborate mathematical theorems, but that ultimately the numbers and all mathematics are merely mental constructions. They further argued that in higher mathematics, any proofs that were not explicitly constructed but which assume things with logic, or refer to unspecified collections, or prove by showing opposites to be impossible, *etc.* are all invalid and are therefore tantamount to mystical or theological statements. Intuitionism did not last very long and gained few supporters in the mathematical community[17].

The major contender of logicism was *formalism* which argued that although there seemed to be something intrinsically basic and vital about the natural numbers, operations involving these and all higher mathematics were really only an exercise in manipulating symbols. David Hilbert[18] was the last great mathematician who was able to

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[16] The absurdity was too much for Russell and he became a recluse during these years, embarrassed to tell people that he was pouring his considerable gifts into what seemed like a childishly simple proposition. Also, the not having sets inside other sets thing didn't work. In fact, the whole self-referential aspect of the "this statement is false" kind of paradoxes turned out not to be a weird quirk that got in the way of serious reasoning; instead it was later discovered that self-reference was of cardinal importance to the basis of mathematics and science. *Cf.* Jacob Bronowski's various collections of essays.

[17] It was mainly propagated by one Dutch guy, LEJ Brouwer (1881–1966) who was brilliant, mystical and, sadly, extremely unwell. He spent most of his later years in a state of crippling paranoia and became a recluse — an oddly common outcome for top mathematicians. However, as Wallace advises us, we shouldn't be too hasty in fitting these kinds of characters into a set of cliches about madness–genius dichotomies. Even more sadly, of course, David Foster Wallace himself committed suicide in 2008. "Great wits are sure to madness near alli'd; / And thin partitions do their bounds divide" — Dryden.

[18] David Hilbert (1862–1943) was a pretty good guy who opposed the purging of Jews from the academy after he retired and the Nazis came to power.

contribute significantly across almost the whole of the discipline[19] and was therefore well placed to articulate and generalise the formalist doctrine, which aimed at systematising all of mathematics, where logicism had aimed to establish the underlying logic of such a system. Hilbert argued that the rules of mathematical deduction were already rigorous and that mathematical proofs were fine, as long as they could ultimately be reducible to arithmetic and algebraic operations. This may seem similar to logicism and indeed both programs were aiming at similar goals of completeness and consistency; but the crucial difference is that formalism held that mathematical statements contain no content and are merely games involving symbols that follow certain rules. This means that if one were to substitute in new symbols and change some rules, one could develop an alternative mathematics, which, according to formalist principles, would be just as valid as the incumbent system and indeed Hilbert held this to be true.

By the 1930s there were many mathematicians working on fortifying the bases of logicism and formalism and much more interest from the philosophical community, especially owing to the work of Russell. But this interest included that from Ludwig Wittgenstein, whose masters thesis[20] of 1919 was gradually accepted as being the beginning of a new era in philosophy and, incidentally, a shocking undermining of Russell's PM, although this was on metaphysical grounds and this did not necessarily permeate the mathematical culture at all. The point was moot because in 1931 the Austrian mathematician Kurt Gödel, aged 25, produced what is possibly the most profound paper ever published in mathematics, logic, or philosophy. "Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme"[21] shattered the basic objective of PM, showing that not only was it impossible for PM to ever prove the soundness of its axioms from within its own rules, but no formal system would ever be able to do so. In other words, any formal system — like standard mathematics, one of Hilbert's alternative mathematics, first order logic, predicate logic[22] — will be either complete or consistent, but not both. So if a formal system is consistent, as Hilbert hoped for mathematics generally and Russell intended with PM, then it has to have certain axioms that simply have to be taken as given, that themselves have no justification using the logic of the system and which are therefore unprovable. A system can get around this only by having some statements which prove its own axioms[23], but in doing so it will necessarily have internal contradictions and will therefore be inconsistent[24].

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[19] It has since become too large and too specialised.

[20] Later published as *Tractatus Logico Philosophicus*.

[21] "On formally undecidable propositions of *Principia Mathematica* and related systems".

[22] Whatever, they're just different kinds of logical systems involving slightly different rules and axioms.

[23] I'll try and explain how he did this below.

[24] The profundity of this result cannot be overstated. For intrinsic logical reasons we will never be able to construct a complete and consistent system to establish objective truth, to axiomatise mathematics or to underpin natural science. Self-reference is again part of the problem. It's also a satisfying nail in the

A critical feature of formal systems that leaves them prey to this fault is self-reference. It is necessary for formal systems to refer to themselves otherwise they become largely useless. A natural language is extremely efficient because we can easily refer to aspects of the language from within it[25]. Mathematics also needs to do this but Russell's theory of types sought to ban mathematical statements that refer implicitly to other mathematical statements, fearing that self-referential paradoxes were insurmountable. In contrast, Gödel embraced self-reference and his method of showing incompleteness was to contrive an elaborate proof to expose the necessity of self-reference. By assigning to every mathematical statement[26] a unique number (almost like a phone number) he was able to construct a mirror system of numbers and then make various statements about *that* system which were true, but which were in contradiction to the original mathematical system because they were about that original system[27].

The immediate implications of these results were to destroy PM and to discredit any kind of formalist program. The contention that mathematical objects have no abstract existence was held by all three schools but Gödel's incompleteness theorems seemed to suggest that mathematical objects have an existence independent of human minds and that mathematical proofs are waiting "out there" to be discovered. His theorems also imply that mathematics as we practise it is imperfect; it is this lack of perfection which, surprisingly, led him to postulate a similarity between our knowledge of physical objects and our knowledge of mathematical objects. In both cases our knowledge is fallible and can be improved upon; in both cases the objects under scrutiny are not reducible to mental entities (Gödel presumes they exist independent of human minds); and in both cases the objects were not constructed by humans. Therefore, a Gödelian would claim that the mathematical objects are just as real as physical objects and further that the shattering of formal systems by Gödel's theorems demonstrates that there is a more perfect world of mathematical forms to which we can't gain access; how

coffin for any outmoded idea that we can have a total explanation for things or that people can obtain absolute certainty — they certainly can't.

[25] Self-reference is standard in spoken and written language. It's trivially easy to make a statement that refers to another statement; like this footnote for example.

[26] That includes every number but also every expression like 1+1=2.

[27] This is so hard to explain but think about it like this: everyone so far has tried to demonstrate the integrity of mathematics by using mathematics, but how can that possibly work without self-reference? And in fact knowing that you have to use the system to justify the system, of course it will always be incomplete… or inconsistent. Bronowski has some good summaries of Gödel's work, but Hofstadter's work in the bibliography is a signal achievement of writing. *Gödel, Escher Bach* is a freakish synthesis of maths, computer science, genetics, code-breaking, linguistics, pattern-recognition, musicology all written in a Lewis Carroll style playfulness and all in the service of investigating how artificial intelligence might be created. It's a cult-classic for a reason and includes some very subtle analogies for explaining Gödel's work.

else would nature operate if not by some consistent and complete logic, beyond human comprehension? How else but by according to the logic of God[28]?

The period after Gödel is in some ways like a reformation of the situation at the end of the nineteenth century. We now have several schools, but with highly reduced aims. Whereas the likes of Hilbert, Russell and Brouwer were attempting comprehensive articulations of what mathematics is and how it can be done, while dismissing any kind of platonist fancy, the later twentieth century has fictionalists, structuralists, nominalists and radical platonists all trying to find some way of accommodating the systemic limitations raised by Gödel with the continued effusion of increasingly sophisticated and abstract mathematical theorems.

Structuralism holds that mathematics is not really about mathematical theorems or indeed elementary mathematical objects (such as numbers), but about the structures which delimit how these elements may be expressed, combined, derived, *etc*. One example that lends weight to this view is that there are, using set theory, two possible methods for deriving the natural numbers[29], both equally plausible yet mutually exclusive. Therefore, according to structuralist accounts, both methods are different systems which make manifest the same mathematical structure. Structures in this sense exist independent of any particular mathematical investigations and independent of mathematicians: they are abstract objects and thus structuralism is currently the most respected platonist theory and seems to accomodate everything Gödel postulated.

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[28] Or should that be Göd? The fact that humans can understand how logical systems can be faulty also suggests that humans can in some sense out-think mathematics or computers. Some people think this shows we have some kind of link to the platonic world of pure forms. In fact, some (especially the physicist Sir Roger Penrose) think that our ability to comprehend maths is the only link we have to the higher realm of perfect forms.

[29] We saw earlier how this was one of the things Russell and other set theory mavins were working on; turns out there's more than one way to get the natural numbers out of sets. If you're interested, here they are. First off the notation for sets is like this: {1,2,3,4,...}. That would be the set of natural numbers. The crucial point here is that if we have a completely empty set it would look like this: { } But we can also denote it with the symbol Ø (some Danish letter) so that Ø represents the null or empty set, in other words Ø = { }. Because sets can contain other sets (remember Russell's theory of types didn't fly) then we can take the empty set and put it inside a set: {Ø}. And then we can take this set and put it inside another like so: {{Ø}}. We can continue this process and as we add each successive layer of nested sets, we can have them represent the natural numbers so Ø = 0; {Ø} = 1; {{Ø}} = 2; {{{Ø}}} = 3; *etc*. But we can do it a slightly different way where we say Ø = 0; {Ø} = 1; {Ø, {Ø}} = 2; {Ø, {Ø}, {Ø, {Ø}}} = 3. The difference is that in the first method each successive number is defined as being one more than the previous. In the latter method it is defined as containing all the numbers that are smaller than it. Both of them make mathematical sense and both make some intuitive sense. The second one is right if we ask ourselves, "Does 3 contain 2 and 1 and 0?" The answer is clearly yes, all those numbers are subsets of the larger one. By now this is getting tedious but here's the catch: is 2 an element of the set representing 3? In the first method, no, in the second, yes. These two statements can't coincide, meaning that if they are isolated they work fine and can be used to build up all the rules of arithmetic and, in theory, higher mathematics; but when examined on a meta level, they are in contradiction, meaning that there is at the heart of set theory a basic epistemological problem, where one cannot decide between alternative statements' truth values. This was pointed out in the '70s by a guy called Paul Benacerraf (1931–).

Taking the place of earlier theories that sought to banish platonism is the contemporary *nominalist* approach which claims that mathematics is really just a set of names that we attribute to certain entities, a particular language that uses numbers and symbols to express relations between (real) objects. To satisfy this claim, nominalists attempt to show that one can dispense with mathematics and still describe everything we need to in the natural sciences, thereby showing that mathematics is just one of several naming schemes or languages we could use to describe relations among things and not some kind of special abstract realm. Hartry Field produced an extraordinary work that did this with Newtonian mechanics. By adopting the stance that the spatial continuum is infinitely divisible, he demonstrated that these concrete locations in space can act as surrogates for the all real numbers and can therefore be substituted into Newtonian mechanics in place of numbers. At its root, Newtonian mechanics really only requires real numbers and functions on them to build up the rest of the theory. Field's attempt has been seen as successful, as far as it goes, with two key caveats that remain undecidable: 1) if nature is not continuous, if the implications of quantum mechanics are generally true, then in fact space is discretised into a finite amount of quanta and so cannot act as surrogates for the infinite set of real or even natural numbers; and 2) it is not at all clear that we can have direct knowledge of all of spacetime, whether it is finite or infinite and therefore it is not obvious that the locations in spacetime are any less abstract than the numbers of which they stand in lieu.

Fictionalism is another, more picturesque, attempt at refuting platonism, by claiming that mathematical theorems are equivalent to fictional stories; objects like rings, sets, numbers, *etc.* exist in the same way that Sherlock Holmes or 221B Baker Street exists. Different people can share an understanding of these fictional entities but they clearly do not exist in the way that physical objects do. Fictionalism fails mainly in the same way that formalism did, *i.e.* that mathematicians do not simply manipulate the symbols in mathematics with complete whimsy, rather they are open to severe scrutiny and elaboration of mathematical ideas in one direction seems to make much more internal sense than another; often they seem to make external sense too.

Indeed the recurrent question lurking beneath all the above debates is whether these various theories accord with what we learn via natural science. The pythagoreans were scandalised when they found, through investigation of real world examples, that the hypotenuse of a right angle triangle with its short sides measuring one unit each, is the square root of two: an irrational — and, to them, profane — number. The discoveries of Einstein in the early twentieth century showed that spacetime itself is best described using non-Euclidean geometries, long thought to be mathematical curios of abstract worlds remote from ours. Even Gödel's incompleteness theorem which appeared to destroy the foundations of the entire mathematical-logical system is indispensable to computer science where self-reference, recursion and incompleteness are necessary for computation and primitive computer languages[30]. It is not surprising, then, that we find the community of philosophers of mathematics somewhat in a state of waiting, as more and more interesting results come back from the vanguard of string theory, black hole astronomy, cosmology and particle physics. Some of these results have given rise to a perspective that takes the universe itself to be *wholly* mathematical.

Radical platonism was aided by the recent work of cosmologist Max Tegmark[31], who argues that we live in a mathematical universe where mathematics is the universal language to describe natural processes around us and that this is because everything in the universe, including us, quite literally *is* a mathematical structure. Some mathematical structures are more complex than others. At the smallest level, nature is not directly observable and therefore our descriptions of subatomic particles really are only mathematical. Certain arrays of subatomic particles can be in very complicated relations to one another and manifest at the macroscale as living organisms. That these sub-structures are self-aware and can think about the very mathematics that instantiate them provides, for some, a pleasing fulfillment of Gödel's priority of self-reference. Critics of the mathematical universe hypothesis, however, point out that it requires a cosmos of infinite universes in which all possible mathematical alternatives are realised; such a theory of multiverses is gaining popularity among cosmologists, but is currently little more than a mathematical speculation[32].

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[30] Following Gödel, Alan Turing (1912–54) took up the work on meta-mathematics and looked at how we could come up with a way to decide whether a mathematical statement was decidable ahead of actually getting on with the arduous task of working it out. This still has massive implications for computing, in terms of what problems it's possible for a computer to solve. Alas, even though Turing was instrumental in cracking the Nazi's communication codes during WWII, he was forsaken by his government and arrested on a charge of sodomy. He was chemically castrated as punishment and died, probably from suicide, but questions are still being asked.

[31] Max Tegmark's (1967–) outré theories are very entertainingly covered in Jim Holt's book (see Bibliography). He doesn't look at Tegmark in particular but does talk to a few other physicists who have pretty full-on religious attitudes towards the mathematical universe. In fact, Holt's book is one of the best non-fiction books I've ever read, mixing serious intellectual pursuit with personal reflections that enhance the story, not those shitty magazine style personal reflections where you learn about what the interviewee's apartment looks like as though it were some kind of useful insight into their work and not just an extremely facile method for making paltry generalisations about a subject.

[32] For those playing at home, here's a quick recap of our isms.

Logicism (maths is all about basic logical axioms) got smashed by Gödel's complete/consistent paradox.

Formalism (it's all just symbols following rules) went the same way.

Intuitionism (other than intuitive counting numbers, maths is just mysticism) didn't get far mainly because it couldn't accommodate higher mathematics.

More recently *structuralism* (maths is about larger structures, more than one way to skin a cat) emerged to cope with Gödel but it is thoroughly platonist.

Nominalism (maths is a just a language we use — physical scientists like this one) is attractive but struggling to cope with more elaborate maths.

Fictionalism (maths is like Sherlock Holmes) isn't really convincing anybody because maths makes more internal sense that Conan Doyle.

The issue of incompleteness is also problematic. Any serious physical theory of how the universe works will need to include some recursively enumerable[33] method for deriving at least the natural numbers and therefore such a system will be prey to Gödel's theorems, leaving the axioms of physics incomplete or inconsistent; does this also imply that the physical system to which they correspond is also incomplete or inconsistent? What would that even mean? And if mathematical objects are real but not physical, then what does that mean for other non-physical ideas such as concepts, names, fictional entities[34] and information?

Somewhat strangely, the modern observer finds herself in an incomplete or even inconsistent philosophical position: even a concerted effort at dispensing with any kind of mysticism, groundless speculation, or argument by appeal to aesthetics leaves her unable to maintain any kind of anti-platonic stance regarding the existence of mathematical objects. The results of mathematics are too extensive and too unusual, it seems, to have them simply be a kind of language which describes nature. And yet they are not perfect either and — if nothing else — are yet to be fully elaborated: large prizes are offered for answering important mathematical problems, some of which date to antiquity. So while structuralism, at present, seems to make the most mathematical sense and appears to be free of gross contradictions, it seems to ably fit a bold conclusion like Tegmark's which says we live in mathematical world. But the stubborn materialist will want to know why it is that we seem to live in a world made up of stuff and substance, not numbers and functions, which latter seem to be the map but not the territory.

Indeed this is where the philosophy of mathematics has been so profound in recent thought, providing the most unusual and challenging case of seemingly abstract objects that, unlike names, concepts, gods, *etc*., really do seem to be entangled with the physical world. That is why it is an area worth studying, because it may be that a better knowledge of the existence of circles, sets and numbers will improve our understanding of the existence of all things, including ourselves. Thus we wait in happy ignorance for advances in computing and cosmology to provide new perspectives on whether the universe is the territory and maths the map, or if the universe itself is mathematical[35].

Radical platonism (the universe *is* maths) is popular with some cosmologists and computer scientists but the jury's out.

[33] Like the way we derived numbers from sets in an iterative sort of process.

[34] Including all gods and other hobgoblins.

[35] Stephen Wolfram (1959–) is a billionaire, genius, megalomaniacal super-nerd. He was one of those freaks who graduated from Oxford when he was five or something and then became a dotcom rich guy and invented the incomparable programming language Mathematica. He also made WolframAlpha.com but found time to write an incredible book called *A New Kind of Science* which lives up to its titular arrogance and his prenominate megalomania. It's included here because in it he purports to solve virtually every outstanding problem in science, mathematics, philosophy and human society all with recourse to what he learned about some extremely basic computer programs called cellular automata.

Wolfram tested alternatives of these simple two-dimensional graphic programs which look like arrays of black or white squares and which follow simple rules (*e.g.* if black square next to left of white square, change colour). Some of the rules generated all black or all white results, some yielded checkerboard or similar patterns, but a few rules produced irregular or highly complex patterns. From these humble little programs Wolfram generalised a theory of computational equivalence. Wolfram's website explains: "the principle of computational equivalence says that systems found in the natural world can perform computations up to a maximal ('universal') level of computational power, and that most systems do in fact attain this maximal level of computational power. Consequently, most systems are computationally equivalent. For example, the workings of the human brain or the evolution of weather systems can, in principle, compute the same things as a computer. Computation is therefore simply a question of translating inputs and outputs from one system to another."

What this lengthy quotation is getting at is that all complex processes in nature, living systems, computer programs, the brain and the universe as a whole are ultimately reducible to simple, local rules (like cellular automata) and that even the greatest complexity is merely an expression of the iterative, cumulative computation performed by the universe's particles moment to moment. Note that if Wolfram is right, then the universe is discrete both spatially and temporally and mathematics is merely one example of a set of programs that can be run and with enough of them concatenated together one can achieve a lot of computational power. *But* the flipside of Wolfram's theories is that many processes are computationally irreducible, meaning that you can't guess ahead of time whether something is computable or what the result is — you just have to run the program. This explains the surprising patterns in some of his cellular automata and the presence of chaotic processes in nature like weather and pretty much anything else we currently cannot predict accurately… which is pretty much everything.

Regarding the central question of this essay, Wolfram is avowedly on the side that says our current mathematics is an historical artefact, rather than a glimpse of a perfect world of forms. But whereas nominalists may well say mathematics is a series of arbitrary symbols we use as a language, Wolfram says that our current mathematics, reducible as it is to a couple of pages of axioms, is merely one formal system in the space of all possible formal systems. In other words, we can generate all the other possible mathematicses by changing the permutations of symbols in those axioms and thereby see that our mathematics is only one of hundreds of thousands; this space also includes formal logic and several other formal systems that produce theorems based on simple axioms. The mindblowing implications of this experiment are that there are whole worlds of mathematics that are yet to be described, all internally consistent (though not necessarily as rich as ours) and that we have settled on our maths because we have generally explored maths hand-in-hand with the physical sciences or real world applications, leading us to follow paths that seem intuitive; *but* we get glimpses of the whole space of mathematics when we run up against unsolvable or hard to solve problems. Wolfram thinks that most mathematical questions would be like this but that we don't randomly select the questions we ask; instead we proceed from problem to problem and end up carving a certain path through largely solvable problems in our historically determined mathematical subspace.

If I was better at this I would have woven his ideas into the main body, but it's not clear where he fits in and philosophers haven't really dealt with his views. His lack of intellectual modesty has worked against him. I hate prediction, so my only one in this essay is that Wolfram's idas will age well in the next few decades.

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